ULTRAHYPERFUNCTIONAL APPROACH TO NON-COMMUTATIVE QUANTUM FIELD THEORY

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Abstract. In the present paper, we intent to enlarge the axiomatic framework of non-commutative quantum field theories (QFT). We consider QFT on non-commutative spacetimes in terms of the tempered ultrahyperfunctions of Sebastião e Silva corresponding to a convex cone, within the framework formulated by Wightman. Tempered ultrahyperfunctions are representable by means of holomorphic functions. As is well known there are certain advantages to be gained from the representation of distributions in terms of holomorphic functions. In particular, for non-commutative theories the Wightman functions involving the $\star$-product, $\mathfrak{W}^\star_m$, have the same form as the standard form $\mathfrak{W}_m$. We conjecture that the functions $\mathfrak{W}^\star_m$ satisfy a set of properties which actually will characterize a non-commutative QFT in terms of tempered ultrahyperfunctions. In order to support this conjecture, we prove for this setting the validity of some important theorems, of which the CPT theorem and the theorem on the Spin-Statistics connection are the best known. We assume the validity of these theorems for non-commutative QFT in the case of spatial non-commutativity only.

Dedicated to Prof. Olivier Piguet on the occasion of his 65th birthday.

1. Introduction

In recent years, many novel questions have emerged in theoretical physics, particularly in non-commutative quantum field theories (NCQFT), for which a considerable effort has been made in order to clarify structural aspects from an axiomatic standpoint [1]-[6]. Axiomatic Quantum Field Theory is the program, originally conceived by Gårding and Wightman [7]-[10], that aims to study of unified form the fundamental postulates, and their consequences, of the two pillars apparently opposite of the modern physics: the Relativity Theory and the Quantum Mechanics. The standard formulation of the axioms of quantum field theories is best expressed by the so-called Wightman axioms, which can be summarized as follows: (I) *Quantum mechanical postulates*. The states are described by vectors of a Hilbert space $\mathscr{H}$. In $\mathscr{H}$, there exists a unitary representation of the Poincaré group, whose translation group admits the closed forward light cone $\overline{V}_+ = \big\{p_\mu \in \mathbb{R}^4 \mid p^2 \geq 0, p^0 \geq 0\big\}$ as its spectrum. There is a unique vaccum state $|\Omega_o\rangle$ in $\mathscr{H}$, which is the unique

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state invariant by translations (this implies in the uniqueness of the vacuum). (II) *Special relativity postulates.* The fields transform covariantly under Poincaré transformations. The microcausality condition imposes that the fields either commute or anti-commute at spacelike separated points $\big[\Phi(x),\Phi(x')\big]_{\pm}=0$ for $(x-x')^2<0$. (III) *Technical postulate.* The assumption of a character of distribution takes essential place among the basis postulates of quantum field theory. In a mathematical language, there are some reasons to consider the fields as *tempered* distributions [7]-[10]. This choice is connected with a definition of *local* properties of distributions. It turn out that all these postulates can be fully reexpressed in terms of an infinite set of tempered distributions, called Wightman distributions (or correlation functions of the theory).

By a variety of reasons, the Wightman framework of local QFT turned out to be too narrow for theoretical physicists, who are interested in handling situations involving in particular NCQFT. One of the reasons is that the commutation relations for the *non-commutative coordinates* $[x_\mu,x_\nu]=i\theta_{\mu\nu}$ break down the Lorentz group $SO(1,3)$ to a *residual* symmetry $SO(1,1)\times SO(2)$. This happens because the deformation parameter $\theta_{\mu\nu}$ is assumed to be a *constant* antisymmetric matrix of lenght dimension two. Although an axiomatic formulation has been proposed based in the *residual* symmetry $SO(1,1)\times SO(2)$ [1]-[6], a serious inconvenient arises of this analysis: the subgroup $SO(1,1)\times SO(2)$ does not allow that particles be classified according to the 4-dimensional Wigner particle concept [11]-[13].

Another reason why the framework of local QFT turned out to be too narrow, it is that NC-QFT are *nonlocal.* This can have implications on highly physical properties. For example, in the formulation of general properties of a field theory the localization plays a fundamental role in the concrete realization of the locality of field operators in coordinate space and spectral condition in energy-momentum space, which are achieved through the *localization of test functions* – the fields are considered tempered functionals on the Schwartz's test function space, the space of rapidly decreasing C^∞-functions. However, the nonlocal character of the interactions in NCQFT seems to indicate that fields are not tempered. In fact, as it was emphasized in [1], the existence of hard infrared singularities in the non-planar sector of the theory, induced by uncancelled quadratic ultraviolet divergences, can destroy the *tempered* nature of the Wightman functions. Besides, the commutation relations $[x_\mu,x_\nu]=i\theta_{\mu\nu}$ also imply uncertainty relations for spacetime coordinates $\Delta x_\mu\Delta x_\nu\sim\big|\theta_{\mu\nu}\big|$, indicating that the notion of spacetime point loses its meaning. Spacetime points are replaced by cells of area of size $\big|\theta_{\mu\nu}\big|$. This observation has led physicists to suggest the existence of a finite lower limit to the possible resolution of distance. Instead, the nonlocal structure of NCQFT manifests itself in the *delocalization* of the interaction regions, which spread over a spacetime domain whose size is determined by the existence of a *minimum length* ℓ_θ related to the scale of nonlocality $\ell_\theta\sim\sqrt{\theta}$ [14]. Among other things, the existence of this *minimum length*

renders impossible the preservation of the local commutativity condition, so it is unclear why we should even consider the microcausal condition based on local fields as in [1, 2, 15].

These are some very important evidences to expect that the traditional Wightman axioms must be somewhat modified within the context of NCQFT [16]. From our point of view, the spacetime non-commutativity can be accommodated simply by choosing a space of generalized functions different from the usual space of Schwartz's tempered distributions. As a matter of fact, in a fundamental formulation of QFT, the mathematical problem can be seen as a problem of the choice of the *right* class of generalized functions which is appropriate for the representation of quantum fields. Thus, the class of generalized functions which one should use in the formulation of NCQFT remains an open problem still to be fully understood.

Some attempts have been made to extend the framework fomulated by Wightman for NCQFT, so as to include a wider class of fields [3, 6]. It has been suggested that NCQFT must should be formulated in terms of generalized functions over the space of analytic test functions S^0 [22]-[28], exploring some ideas by Soloviev to nonlocal quantum fields [25]-[28].[1] In this case, the fields are so singulars that, of course, one of the conceptual problems we are faced is find an adequate generalization of the causality condition. Soloviev has suggested to replace the ordinary causality condition by an asymptotic causality condition. Despite its apparent weakness, the asymptotic causality condition in the sense of Soloviev yet one allows us to show the validity of the CPT theorem and the Spin-Statistic connection for NCQFT [3]. And more, the existence of a Borchers class for a non-commutative field is shown [4]. On the other hand, recently, different definitions of perturbative theory to NCQFT [31, 32] seem to point out that the nonlocal interactions in NCQFT improve the UV behavior of theory. It is therefore reasonable to consider another space of test functions where the fields are not highly singulars as adopted in [3, 6].

In this paper, we present an alternative approach. Because NCQFT suggest the existence of a minimum length ℓ_θ, we will assume as space of test functions for NCQFT the space $\mathfrak{H}$ of rapidly decreasing entire functions in any horizontal strip. The elements of the dual space of the space $\mathfrak{H}$ are so-called tempered ultrahyperfunctions [33]-[48] and have the advantage of being representable by means of holomorphic functions. Tempered ultrahyperfunctions generalize the notion of hyperfunctions on $\mathbb{R}^n$ but *can not* be localized as hyperfunctions. Because of this, NCQFT of this sort will be called *quasilocal*, namely, the fields are localizable only in regions greater than the scale of nonlocality ℓ_θ. We shall walk along the general lines proposed recently by Brüning-Nagamachi [46]. They have conjectured that tempered ultrahyperfunctions, *i.e.*, those ultrahyperfunctions which

[1]More recently, Chaichian *et al* [29] have obtained a result that the appropriate space of test functions in the Wightman approach to non-commutative quantum field theory is one of the Gel'fand-Shilov spaces S^β, with $\beta < 1/2$ [30]. The authors of Refs. [3, 6] assume $\beta = 0$ in order to emphasize that this is the smallest space among the Gel'fand-Shilov spaces S^β traditionally adopted in nonlocal quantum field theory, as indicated from non-commutative quantum field theory.

admit the Fourier transform as an isomorphism of topological vector spaces, are well adapted for their use in quantum field theory with a fundamental length. In particular, we shall consider tempered ultrahyperfunctions in a setting which includes the results of [33, 34, 35] as special cases, by considering functions analytic in tubular radial domains [41, 47, 48]. We shall denote the NCQFT in terms of tempered ultrahyperfunctions by UHFNCQFT for brevity, hereafter.

The presentation of the paper is organized as follows. In Section 2, for the convenience of the reader, we present the reasons why tempered ultrahyperfunctions are well adapted for their use in NCQFT, going through a simple example taken from Ref. [46]. Section 3 contains an exposition of the theory of tempered ultrahyperfuntions, where we include and prove some results which are important in applications to quantum field theory. Section 4 is devoted to the formulation of the axioms for UHFNCQFT in terms of the Wightman functionals. How the properties of the Wightman functionals change when we pass to the test function space which are entire analytic functions of rapid decrease in any horizontal strip is considered. In Section 5, we derive for our UHFNCQFT the validity of some important theorems, obtained previously for essentially nonlocalizable fields [3, 4, 6]. These include the existence of CPT symmetry and the connection between Spin and Statistics for UHFNCQFT. Throughout the paper we assume only the case of space-space non-commutativity, *i.e.*, $\theta_{0i} = 0$, with $i = 1, 2, 3$. It is well known that if there is space-time non-commutativity, the resulting theory violates the causality and unitarity [49, 50]. For most our purposes, we consider for simplicity a theory with only one basic field, a neutral scalar field. Section 6 is reserved for our concluding remarks.

2. Motivation

For the sake of completeness in the exposition, we recall the example which has motivated Brüning-Nagamachi [46] to conjecture that tempered ultrahyperfunctions are suitable in order to treat quantum field theories with a minimum length. Consider the Dirac delta measure $\delta(x + a)$, which when applied to a continuous function $f(x)$ produces the value $f(-a)$

$$\int \delta(x + a) f(x)\, dx = f(-a)\ .$$

By using a generalization of the Cauchy's integral formula, we define $\delta(x + a)$ applied to a holomorphic function $f(z)$ on an open set $\Omega \subset \mathbb{C}$. Assuming that $0 \in \Omega$ and letting $\gamma = \partial\Omega$ denote the boundary of Ω, we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z + a}\, dz = f(-a)\ , \quad \text{for } z \in \Omega\ . \tag{2.1}$$

Define $\mathfrak{H}(T(-\ell, \ell))$ as being the space of all holomorphic functions $f(z)$ on $T(-\ell, \ell) = \mathbb{R}^n + i(-\ell, \ell) \subset \mathbb{C}$. In this case, from (2.1), for $f(z) \in \mathfrak{H}(T(-\ell, \ell))$ and $|a| < \ell$, $f(-a)$ can be given by the Taylor's

series of center in zero

$$f(-a) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} f^{(n)}(0) \ .$$

This series possesses the functional representation

$$F(f) = \int \Big[\sum_{n=0}^{\infty} \frac{a^n}{n!} \delta^{(n)}(x)\Big] f(x)\ dx = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} f^{(n)}(0)$$

$$= f(-a) = \int \delta(x+a) f(x)\ dx \ .$$

Thus, as an equation for functionals defined on the function space $\mathfrak{H}(T(-\ell,\ell))$, we have the identification

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} \delta^{(n)}(x) = \delta(x+a) \ ,$$

in the distributional sense. In other words, the sequence of generalized functions

$$S_N = \sum_{n=0}^{N} \frac{a^n}{n!} \delta^{(n)}(x) \ ,$$

with support $\{0\}$ weakly converges to the generalized function $\delta(x+a)$ with support $\{-a\}$, as $N \to \infty$. However, if $|a| > \ell$, this sequence does not converge in the dual space of $\mathfrak{H}(T(-\ell,\ell))$.

The motivation for suggesting that tempered ultrahyperfunctions are well adapted for their use in quantum field theory with a fundamental length lies in the following fact: the non-local structure of the functional F is represented by a dislocation of the support from $\{0\}$ to $\{-a\}$. According to Brüning-Nagamachi [46], this means that, if $|a| < \ell$, then the elements in the dual space of $\mathfrak{H}(T(-\ell,\ell))$ do not distinguish between the points $\{0\}$ to $\{-a\}$, but if $|a| > \ell$ the elements in $\mathfrak{H}'(T(-\ell,\ell))$ can distinguish between the points $\{0\}$ to $\{-a\}$. Since $|a| < \ell$ is arbitrary, one can say that the elements in $\mathfrak{H}'(T(-\ell,\ell))$ distinguish points only in spacetime regions large in comparison with ℓ. This is the reason why we discuss here a mathematically more satisfactory approach for NCQFT. The tempered ultrahyperfunctions have this property.

Remark 1. Such an example was already considered in 1958 by Güttinger [51] in order to treat certain exactly soluble models which would correspond to field theories with non-renormalizable interactions.

3. Tempered Ultrahyperfunctions

The interest in tempered ultrahyperfunctions arose simultaneously with the growing interest in various classes of analytic functionals and various attempts to develop a theory of such functionals which would be analogous to the Schwartz theory of distributions. Tempered ultrahyperfunctions

were first introduced in papers of Sebastião e Silva [33, 34] and Hasumi [35] as the strong dual of the space of test functions $\mathfrak{H}$ of rapidly decreasing entire functions in any horizontal strip. As a matter of fact, these objects are equivalence classes of holomorphic functions defined by a certain space of functions which are analytic in the 2^n octants in $\mathbb{C}^n$ and represent a natural generalization of the notion of hyperfunctions on $\mathbb{R}^n$, but are *non-localizable*. In this section, we recall some basic properties of the tempered ultrahyperfunction space which are the most important in applications to quantum field theory.

To begin with, we shall define our notation. We will use the standard multi-index notation. Let $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) be the real (resp. complex) n-space whose generic points are denoted by $x = (x_1, \ldots, x_n)$ (resp. $z = (z_1, \ldots, z_n)$), such that $x + y = (x_1 + y_1, \ldots, x_n + y_n)$, $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$, $x \geq 0$ means $x_1 \geq 0, \ldots, x_n \geq 0$, $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ and $|x| = |x_1| + \cdots + |x_n|$. Moreover, we define $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_o^n$, where $\mathbb{N}_o$ is the set of non-negative integers, such that the length of α is the corresponding ℓ^1-norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha + \beta$ denotes $(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$, $\alpha \geq \beta$ means $(\alpha_1 \geq \beta_1, \ldots, \alpha_n \geq \beta_n)$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, and

$$D^\alpha \varphi(x) = \frac{\partial^{|\alpha|}\varphi(x_1, \ldots, x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \ .$$

Let Ω be a set in $\mathbb{R}^n$. Then we denote by Ω° the interior of Ω and by $\overline{\Omega}$ the closure of Ω. For $r > 0$, we denote by $B(x_o; r) = \big\{x \in \mathbb{R}^n \mid |x - x_o| < r\big\}$ a open ball and by $B[x_o; r] = \big\{x \in \mathbb{R}^n \mid |x - x_o| \leq r\big\}$ a closed ball, with center at point x_o and of radius $r = (r_1, \ldots, r_n)$, respectively.

We consider two n-dimensional spaces – x-space and ξ-space – with the Fourier transform defined

$$\widehat{f}(\xi) = \mathscr{F}[f(x)](\xi) = \int_{\mathbb{R}^n} f(x) e^{i\langle \xi, x\rangle} d^n x \ ,$$

while the Fourier inversion formula is

$$f(x) = \mathscr{F}^{-1}[\widehat{f}(\xi)](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-i\langle \xi, x\rangle} d^n \xi \ .$$

The variable ξ will always be taken real while x will also be complexified – when it is complex, it will be noted $z = x + iy$. The above formulas, in which we employ the symbolic "function notation," are to be understood in the sense of distribution theory.

We shall consider the function

$$h_K(\xi) = \sup_{x \in K} \big|\langle \xi, x\rangle\big| \ , \quad \xi \in \mathbb{R}^n \ ,$$

the indicator of K, where K is a compact set in $\mathbb{R}^n$. $h_K(\xi) < \infty$ for every $\xi \in \mathbb{R}^n$ since K is bounded. For sets $K = \big[-k, k\big]^n$, $0 < k < \infty$, the indicator function $h_K(\xi)$ can be easily determined:

$$h_K(\xi) = \sup_{x \in K} \big|\langle \xi, x\rangle\big| = k|\xi| \ , \quad \xi \in \mathbb{R}^n \ , \quad |\xi| = \sum_{i=1}^n |\xi_i| \ .$$

Let K be a convex compact subset of $\mathbb{R}^n$, then $H_b(\mathbb{R}^n;K)$ (b stands for bounded) defines the space of all functions in $C^\infty(\mathbb{R}^n)$ such that $e^{h_K(\xi)}D^\alpha f(\xi)$ is bounded in $\mathbb{R}^n$ for any multi-index α. One defines in $H_b(\mathbb{R}^n;K)$ seminorms

$$\|\varphi\|_{K,N} = \sup_{\substack{\xi\in\mathbb{R}^n \\ \alpha\leq N}} \left\{ e^{h_K(\xi)} |D^\alpha f(\xi)| \right\} < \infty \ , \quad N = 0,1,2,\dots \ . \tag{3.1}$$

Now, let $T(\Omega) = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ be the tubular set of all points z, such that $y_i = \operatorname{Im} z_i$ belongs to the domain Ω, *i.e.*, Ω is a connected open set in $\mathbb{R}^n$ called the basis of the tube $T(\Omega)$. Let K be a convex compact subset of $\mathbb{R}^n$, then $\mathfrak{H}_b(T(K))$ defines the space of all C^∞ functions φ on $\mathbb{R}^n$ which can be extended to $\mathbb{C}^n$ to be holomorphic functions in the interior $T(K^\circ)$ of $T(K)$ such that the estimate

$$|\varphi(z)| \leq \mathbf{C}(1+|z|)^{-N} \tag{3.2}$$

is valid for some constant $\mathbf{C} = \mathbf{C}_{K,N}(\varphi)$. The best possible constants in (3.2) are given by a family of seminorms in $\mathfrak{H}_b(T(K))$

$$\|\varphi\|_{T(K),N} = \sup_{z\in T(K)} \left\{ (1+|z|)^N |\varphi(z)| \right\} < \infty \ , \quad N = 0,1,2,\dots \ . \tag{3.3}$$

Next, we consider a set of results which will characterize the spaces introduced above.

Lemma 3.1. *If $K_i \subset K_{i+1}$ are two convex compact sets, then the following canonical injections holds: (i) $\mathfrak{H}_b(T(K_{i+1})) \hookrightarrow \mathfrak{H}_b(T(K_i))$, (ii) $H_b(\mathbb{R}^n;K_{i+1}) \hookrightarrow H_b(\mathbb{R}^n;K_i)$.*

Proof. We prove the first item. If $K_i \subset K_{i+1}$ and $\varphi \in \mathfrak{H}_b(T(K_{i+1}))$, then $\varphi \in \mathfrak{H}_b(T(K_i))$. By taking the restriction of $\varphi \in \mathfrak{H}_b(T(K_{i+1}))$ to $T(K_i)$, it follows that

$$\sup_{z\in T(K_{i+1})} \left\{ (1+|z|)^j |\varphi(z)| \right\} = \sup_{z\in T(K_i)} \left\{ (1+|z|)^j |\varphi(z)| \right\} \ .$$

Therefore, the topology induced by $\mathfrak{H}_b(T(K_{i+1}))$ on $\mathfrak{H}_b(T(K_i))$ is identical with the topology of $\varphi \in \mathfrak{H}_b(T(K_i))$. The proof of second statement is similar, taking into account the seminorm (3.1). □

Let O be a convex open set of $\mathbb{R}^n$. To define the topologies of $H(\mathbb{R}^n;O)$ and $\mathfrak{H}(T(O))$ it suffices to let K range over an increasing sequence of convex compact subsets $K_1, K_2, \dots$ contained in O such that for each $i = 1,2,\dots$, $K_i \subset K_{i+1}^\circ$ and $O = \bigcup_{i=1}^\infty K_i$. Then the spaces $H(\mathbb{R}^n;O)$ and $\mathfrak{H}(T(O))$ are the projective limits of the spaces $H_b(\mathbb{R}^n;K)$ and $\mathfrak{H}_b(T(K))$, respectively, *i.e.*, we have that

$$H(\mathbb{R}^n;O) = \lim_{K\subset O} \operatorname{proj} \ H_b(\mathbb{R}^n;K) \ , \tag{3.4}$$

and

$$\mathfrak{H}(T(O)) = \lim_{K\subset O} \operatorname{proj} \ \mathfrak{H}_b(T(K)) \ , \tag{3.5}$$

where the projective limit is taken following the restriction mappings according to the Lemma 3.1.

Remark 2. Any C^∞ function of exponential growth is a multiplier in $H(\mathbb{R}^n;O)$, while that any C^∞ function which can be extended to be an entire function of polynomial growth is a multiplier in $\mathfrak{H}(T(O))$. Besides, the space $H(\mathbb{R}^n;O)$ is continuosly embedded into Schwartz space $\mathscr{S}(\mathbb{R}^n)$, and elements of $\mathscr{S}(\mathbb{R}^n)$ are also multipliers for the space $H(\mathbb{R}^n;O)$ [35].

Lemma 3.2. *The spaces $\mathfrak{H}(T(O))$ and $H(\mathbb{R}^n;O)$ are Hausdorff locally convex spaces.*

Proof. First, we prove that $\mathfrak{H}(T(O))$ is a Hausdorff locally convex space. Let $\{K_i\}_{i=1,2,\dots}$ be the usual increasing sequence of compact subsets of O, whose union is O, and such that with K_i is the closure of its interior, K_{i+1}°; for all i, $K_i \subset K_{i+1}^\circ$. We shall prove that each element of the base for neighborhoods of 0 generated by the open balls

$$\mathfrak{B}_{i,n}(0) = \Big\{\varphi \in \mathfrak{H}(T(K_i)) \mid \|\varphi\|_{T(K_i),j} = \sup_{z\in T(K_i)} \Big[(1+|z|)^j|\varphi(z)|\Big] < n^{-1}, n\in\mathbb{N}\Big\}\ ,$$

contains at least one convex neighborhood of 0. For this, it is sufficient to show that there exist natural numbers ℓ, n' such that $\mathfrak{B}_{\ell,n'}(0) \subset \mathfrak{B}_{i,n}(0)$. In fact, one can always choose ℓ such that $K_\ell \subset K_i$. Then, $\|\varphi\|_{T(K_\ell),j} \leq \|\varphi\|_{T(K_i),j}$ if $n < n'$ and $\ell \leq i$. Now, consider $\|\lambda\varphi_1 + (1-\lambda)\varphi_2\|_{T(O),j}$, with $0 \leq \lambda \leq 1$ and $\varphi_1, \varphi_2 \in \mathfrak{B}_{\ell,n'}(0)$. But,

$$\begin{aligned}\|\lambda\varphi_1 + (1-\lambda)\varphi_2\|_{T(O),j} &\leq \|\lambda\varphi_1\|_{T(O),j} + \|(1-\lambda)\varphi_2\|_{T(O),j} \\ &\leq \lambda\|\varphi_1\|_{T(O),j} + (1-\lambda)\|\varphi_2\|_{T(O),j} \\ &< \lambda n^{-1} + (1-\lambda)n^{-1} = n^{-1}\ .\end{aligned}$$

Hence, $\lambda\varphi_1 + (1-\lambda)\varphi_2 \in \mathfrak{B}_{\ell,n'}(0)$. This proves that $\mathfrak{H}(T(O))$ is locally convex. Now, let $\varphi_1, \varphi_2, \psi \in \mathfrak{H}(T(O))$. Consider that for the pair of distinct functions φ_1, φ_2, $\|\varphi_1 - \varphi_2\|_{T(O),j} = \varepsilon > 0$. Let $\boldsymbol{\phi}(\varphi_i) = \mathfrak{B}_{\varepsilon/3}(\varphi_i) = \big\{\psi \in \mathfrak{H}(T(O)) \mid \|\varphi_i - \psi\|_{T(O),j} < \varepsilon/3, i = 1,2\big\}$. For if $\psi \in \boldsymbol{\phi}(\varphi_1) \cap \boldsymbol{\phi}(\varphi_2)$, we have $\|\varphi_1 - \psi\|_{T(O),j} < \varepsilon/3$ and $\|\varphi_2 - \psi\|_{T(O),j} < \varepsilon/3$. Therefore, it follows that $\varepsilon = \|\varphi_1 - \varphi_2\|_{T(O),j} = \|\varphi_1 - \psi + \psi - \varphi_2\|_{T(O),j} \leq \|\varphi_1 - \psi\|_{T(O),j} + \|\varphi_2 - \psi\|_{T(O),j} < 2\varepsilon/3$, which is a contradiction. Hence, $\mathfrak{H}(T(O))$ is Hausdorff. The proof that $H(\mathbb{R}^n;O)$ is a Hausdorff locally convex space is immediate, by considering that the base for neighborhoods of 0 is generated by the open balls

$$\mathfrak{B}_{i,n}(0) = \Big\{\varphi \in H(\mathbb{R}^n;K_i) \mid \|\varphi\|_{K_i,j} = \sup_{x\in\mathbb{R}^n;\alpha\leq j} \big\{e^{h_{K_i}(\xi)}|D^\alpha f(\xi)|\big\} < n^{-1}, n\in\mathbb{N}\Big\}\ ,$$

and the proof is complete. ∎

Theorem 3.3. *The spaces $\mathfrak{H}(T(O))$ and $H(\mathbb{R}^n;O)$ are Fréchet spaces.*

Proof. That $\mathfrak{H}(T(O))$ is metrizable is clear from Theorem V.5 in [52], if we endow the space $\mathfrak{H}(T(O))$ with the metric $d(\varphi_1,\varphi_2) = \sum_{i=1}^{\infty} a_i\|\varphi_1-\varphi_2\|_{T(O),i}/\big[1+\|\varphi_1-\varphi_2\|_{T(O),i}\big]$, such that $\sum_{i=1}^{\infty} a_i < \infty$. Thus, it remains to show that $\mathfrak{H}(T(O))$ is complete. Let $\{\varphi_n\}$ be a sequence of functions in $\mathfrak{H}(T(O))$. We shall take $\varphi_j \in \{\varphi_n\}$. Given $\varepsilon > 0$, there exists n_o such that for $p \geq n_o$ and $n \geq n_o$, we have $d(\varphi_j,\varphi_n) < \varepsilon/2$ and $d(\varphi_j,\varphi_p) < \varepsilon/2$. Then, it follows that $d(\varphi_p,\varphi_n) \leq d(\varphi_j,\varphi_p) + d(\varphi_j,\varphi_n) < \varepsilon/2+\varepsilon/2 = \varepsilon$. This proves that $\{\varphi_n\}$ is Cauchy and hence $\mathfrak{H}(T(O))$ is complete. Thus $\mathfrak{H}(T(O))$ is Fréchet. For the proof that $H(\mathbb{R}^n;O)$ is Fréchet see [37] (and in the case of $O=\mathbb{R}^n$ see [35]). □

It is an elementary fact that $\mathfrak{H}(T(O))$ and $H(\mathbb{R}^n;O)$ are Banach spaces.

Theorem 3.4 (Brüning-Nagamachi [46], Proposition 2.6)**.** *Let $O \subset \mathbb{R}^n$ be a nonempty convex open subset. Then the spaces $\mathfrak{H}(T(O))$ and $H(\mathbb{R}^n;O)$ are nuclear Fréchet spaces and, in particular, reflexive.*

In light of the Theorems 3.3 and 3.4, it follows that the spaces $\mathfrak{H}(T(O))$ and $H(\mathbb{R}^n;O)$ are barreled [53, Corollary 1, p.347] and quasi-complete [53, p.354]. According to Treves [53, Corollary 3, p.520] and Schaefer [54, exercise 19b, p.194], each quasi-complete barreled nuclear space is a Montel space. Thus, one immediately arrives at

Corollary 3.5. *The spaces $\mathfrak{H}(T(O))$ and $H(\mathbb{R}^n;O)$ are Montel spaces.*

Theorem 3.6 ([35, 37, 46])**.** *The space $\mathscr{D}(\mathbb{R}^n)$ of all C^∞-functions on $\mathbb{R}^n$ with compact support is dense in $H(\mathbb{R}^n;K)$ and $H(\mathbb{R}^n;O)$. Moreover, the space $H(\mathbb{R}^n;\mathbb{R}^n)$ is dense in $H(\mathbb{R}^n;O)$ and in $H(\mathbb{R}^n;K)$, and $H(\mathbb{R}^m;\mathbb{R}^m)\otimes H(\mathbb{R}^n;\mathbb{R}^n)$ is dense in $H(\mathbb{R}^{m+n};\mathbb{R}^{m+n})$.*

Theorem 3.7 (Kernel theorem [46])**.** *Let M be a separately continuous multilinear functional on $[\mathfrak{H}(T(\mathbb{R}^4))]^n$. Then there is a unique functional $F \in \mathfrak{H}'(T(\mathbb{R}^{4n}))$, for all $f_i \in \mathfrak{H}(T(\mathbb{R}^4))$, $i=1,\dots,n$ such that $M(f_1,\dots,f_n) = F(f_1\otimes\cdots\otimes f_n)$.*

Theorem 3.8 ([37, 46])**.** *The space $\mathfrak{H}(T(\mathbb{R}^n))$ is dense in $\mathfrak{H}(T(O))$ and the space $\mathfrak{H}(T(\mathbb{R}^{m+n}))$ is dense in $\mathfrak{H}(T(O))$.*

From Theorem 3.6 we have the following injections [37]: $H'(\mathbb{R}^n;K) \hookrightarrow H'(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n)$ and $H'(\mathbb{R}^n;O) \hookrightarrow H'(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n)$.

Definition 3.9. *The dual space $H'(\mathbb{R}^n;O)$ of $H(\mathbb{R}^n;O)$ is the space of distributions of exponential growth.*

A distribution $V \in H'(\mathbb{R}^n;O)$ may be expressed as a finite order derivative of a continuous function of exponential growth

$$V = D^\gamma_\xi[e^{h_K(\xi)}g(\xi)]\ ,$$

where $g(\xi)$ is a bounded continuous function. For $V \in H'(\mathbb{R}^n;O)$ the following result is known:

Lemma 3.10 ([37]). *A distribution $V \in \mathscr{D}'(\mathbb{R}^n)$ belongs to $H'(\mathbb{R}^n;O)$ if and only if there exists a multi-index γ, a convex compact set $K \subset O$ and a bounded continuous function $g(\xi)$ such that*

$$V = D_\xi^\gamma [e^{h_K(\xi)} g(\xi)] \ .$$

For any element $U \in \mathfrak{H}'$, its Fourier transform is defined to be a distribution V of exponential growth, such that the Parseval-type relation $V(\varphi) = U(\psi)$, $\varphi \in H$, $\psi = \mathscr{F}[\varphi] \in \mathfrak{H}$, holds. In the same way, the inverse Fourier transform of a distribution V of exponential growth is defined by the relation $U(\psi) = V(\varphi)$, $\psi \in \mathfrak{H}$, $\varphi = \mathscr{F}^{-1}[\psi] \in H$.

Proposition 3.11 ([37]). *If $\varphi \in H(\mathbb{R}^n;O)$, the Fourier transform of φ belongs to the space $\mathfrak{H}(T(O))$, for any open convex nonempty set $O \subset \mathbb{R}^n$. By the dual Fourier transform $H'(\mathbb{R}^n;O)$ is topologically isomorphic with the space $\mathfrak{H}'(T(-O))$.*

Let us now recall very briefly the basic definition of tempered ultrahyperfunctions. These are defined as elements of a certain subspace of Z' of ultradistributions of Gel'fand and Shilov which admit representations in terms of analytic functions on the complement of some closed horizontal strip of the complex space, and having polynomial growth on the complement of an open neighborhood of that strip.

Let $\mathscr{H}_{\boldsymbol{\omega}}$ be the space of all functions $f(z)$ such that (i) $f(z)$ is analytic for $\{z \in \mathbb{C}^n \mid |\mathrm{Im}\, z_1| > p, |\mathrm{Im}\, z_2| > p, \ldots, |\mathrm{Im}\, z_n| > p\}$, (ii) $f(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n \mid |\mathrm{Im}\, z_1| \geqq p, |\mathrm{Im}\, z_2| \geqq p, \ldots, |\mathrm{Im}\, z_n| \geqq p\}$, where $p = 0, 1, 2, \ldots$ depends on $f(z)$ and (iii) $f(z)$ is bounded by a power of z, $|f(z)| \leq \mathbf{C}(1 + |z|)^N$, where $\mathbf{C}$ and N depend on $f(z)$. Define the *kernel* of the mapping $f : \mathfrak{H}(T(\mathbb{R}^n)) \to \mathbb{C}$ by $\boldsymbol{\Pi}$, as the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$ (a pseudo-polynomial is a function of z of the form $\sum_s z_j^s G(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$, with $G(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathscr{H}_{\boldsymbol{\omega}}$). Then, $f(z) \in \mathscr{H}_{\boldsymbol{\omega}}$ belongs to the kernel $\boldsymbol{\Pi}$ if and only if $f(\psi(x)) = 0$, with $\psi(x) \in \mathfrak{H}(T(\mathbb{R}^n))$ and $x = \mathrm{Re}\, z$. Consider the quotient space $\mathscr{U} = \mathscr{H}_{\boldsymbol{\omega}}/\boldsymbol{\Pi}$. The set $\mathscr{U}$ is the space of tempered ultrahyperfunctions. Thus, we have the

Definition 3.12. *The space of tempered ultrahyperfunctions, denoted by $\mathscr{U}(\mathbb{R}^n)$, is the space of continuous linear functionals defined on $\mathfrak{H}(T(\mathbb{R}^n))$.*

In the following, we will put $\mathfrak{H} = \mathfrak{H}(\mathbb{C}^n) = \mathfrak{H}(T(\mathbb{R}^n))$ and the dual space of $\mathfrak{H}$ will be denoted by $\mathfrak{H}'$.

Theorem 3.13 (Hasumi [35], Proposition 5). *The space of tempered ultrahyperfunctions $\mathscr{U}$ is algebraically isomorphic to the space of generalized functions $\mathfrak{H}'$.*

3.1. **Tempered Ultrahyperfunctions Corresponding to a Proper Convex Cone.** Next, we consider tempered ultrahyperfunctions in a setting which includes the results of [33, 35, 37] as special cases, by considering analytic functions in tubular radial domains [40, 41, 47, 48] and hence includes the important setting for quantum field theory of tube domains over light cones. All the results below are taken from Refs. [47, 48] and hence the proofs will not be repeated.

We start by introducing some terminology and simple facts concerning cones. An open set $C \subset \mathbb{R}^n$ is called a cone if $x \in C$ implies $\lambda x \in C$ for all $\lambda > 0$. Moreover, C is an open connected cone if C is a cone and if C is an open connected set. In the sequel, it will be sufficient to assume for our purposes that the open connected cone C in $\mathbb{R}^n$ is an open convex cone with vertex at the origin and *proper*, that is, it contains no any straight line. A cone C' is called compact in C – we write $C' \Subset C$ – if the projection $\mathsf{pr}\overline{C}' \stackrel{\text{def}}{=} \overline{C}' \cap S^{n-1} \subset \mathsf{pr}C \stackrel{\text{def}}{=} C \cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in x-space, we associate with C a closed convex cone C^* in ξ-space which is the set $C^* = \big\{\xi \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq 0, \forall x \in C\big\}$. The cone C^* is called the *dual cone* of C. By $T(C)$ we will denote the set $\mathbb{R}^n + iC \subset \mathbb{C}^n$. If C is open and connected, $T(C)$ is called the tubular radial domain in $\mathbb{C}^n$, while if C is only open $T(C)$ is referred to as a tubular cone. In the former case we say that $f(z)$ has a boundary value $U = BV(f(z))$ in $\mathfrak{H}'$ as $y \to 0$, $y \in C$ or $y \in C' \Subset C$, respectively, if for all $\psi \in \mathfrak{H}$ the limit

$$\langle U, \psi\rangle = \lim_{\substack{y\to 0\\ y\in C \text{ or } C'}} \int_{\mathbb{R}^n} f(x+iy)\psi(x) d^n x \ ,$$

exists. We will deal with tubes defined as the set of all points $z \in \mathbb{C}^n$ such that

$$T(C) = \Big\{x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C, |y| < \delta\Big\} \ ,$$

where $\delta > 0$ is an arbitrary number.

An important example of tubular radial domain used in quantum field theory is the tubular radial domain with the forward light-cone, V_+, as its basis

$$V_+ = \Big\{z \in \mathbb{C}^n \mid \operatorname{Im} z_1 > \Big(\sum_{i=2}^{n} \operatorname{Im}^2 z_i\Big)^{\frac{1}{2}}, \operatorname{Im} z_1 > 0\Big\} \ .$$

Let C be an open convex cone, and let $C' \Subset C$. Let $B[0;r]$ denote a **closed** ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C';r) = \mathbb{R}^n + i\big(C' \setminus \big(C' \cap B[0;r]\big)\big)$. We are going to introduce a space of holomorphic functions which satisfy certain estimate according to Carmichael [40]. We want to consider the space consisting of holomorphic functions $f(z)$ such that

$$\big|f(z)\big| \leq \mathbf{C}(C')(1+|z|)^N e^{h_{C^*}(y)} \ , \quad z \in T(C';r) \ , \tag{3.6}$$

where $h_{C^*}(y) = \sup_{\xi\in C^*} |\langle \xi, y\rangle|$ is the indicator of C^*, $\mathbf{C}(C')$ is a constant that depends on an arbitrary compact cone C' and N is a non-negative real number. The set of all functions $f(z)$ which

are holomorphic in $T(C';r)$ and satisfy the estimate (3.6) will be denoted by $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$. Throughout the remainder of this paper $T(C';r)$ will denote the set $\mathbb{R}^n + i\big(C' \setminus \big(C' \cap B[0;r]\big)\big)$.

Remark 3. The space of functions $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ constitutes a generalization of the space $\mathfrak{A}_{\omega}^{i}$ of Sebastião e Silva [33] and the space $\mathcal{a}_{\omega}$ of Hasumi [35] to arbitrary tubular radial domains in $\mathbb{C}^n$.

Lemma 3.14 ([41, 47])**.** *Let C be an open convex cone, and let $C' \Subset C$. Let $h(\xi) = e^{k|\xi|}g(\xi)$, $\xi \in \mathbb{R}^n$, be a function with support in C^*, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$. Let y be an arbitrary but fixed point of $C' \setminus \big(C' \cap B[0;r]\big)$. Then $e^{-\langle \xi,y\rangle}h(\xi) \in L^2$, as a function of $\xi \in \mathbb{R}^n$.*

Definition 3.15. *We denote by $H'_{C^*}(\mathbb{R}^n;O)$ the subspace of $H'(\mathbb{R}^n;O)$ of distributions of exponential growth with support in the cone C^*:*

$$H'_{C^*}(\mathbb{R}^n;O) = \Big\{V \in H'(\mathbb{R}^n;O) \mid \operatorname{supp}(V) \subseteq C^*\Big\} . \tag{3.7}$$

Lemma 3.16 ([41, 47])**.** *Let C be an open convex cone, and let $C' \Subset C$. Let $V = D_\xi^\gamma[e^{h_K(\xi)}g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k,k\big]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n;O)$. Then $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi,z\rangle})$ is an element of $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.*

We now shall define the main space of holomorphic functions with which this paper is concerned. Let C be a proper open convex cone, and let $C' \Subset C$. Let $B(0;r)$ denote an **open** ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C';r) = \mathbb{R}^n + i\big(C' \setminus \big(C' \cap B(0;r)\big)\big)$. Throughout this section, we consider functions $f(z)$ which are holomorphic in $T(C') = \mathbb{R}^n + iC'$ and which satisfy the estimate (3.6), with $B[0;r]$ replaced by $B(0;r)$. We denote this space by $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$. We note that $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}} \subset \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ for any open convex cone C. Put $\mathscr{U}_c = \mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}/\boldsymbol{\Pi}$, that is, $\mathscr{U}_c$ is the quotient space of $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$ by set of pseudo-polynomials $\boldsymbol{\Pi}$.

Definition 3.17. *The set $\mathscr{U}_c$ is the space of tempered ultrahyperfunctions corresponding to a proper open convex cone $C \subset \mathbb{R}^n$.*

The following theorem shows that functions in $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$ have distributional boundary values in $\mathfrak{H}'$. Further, it shows that functions in $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$ satisfy a strong boundedness property in $\mathfrak{H}'$.

Theorem 3.18 ([48])**.** *Let C be an open convex cone, and let $C' \Subset C$. Let $V = D_\xi^\gamma[e^{h_K(\xi)}g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k,k\big]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n;O)$. Then*

(i) $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi,z\rangle})$ *is an element of* $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$,

(ii) $\big\{f(z) \mid y = \operatorname{Im} z \in C' \Subset C, |y| \leq Q\big\}$ *is a strongly bounded set in $\mathfrak{H}'$, where Q is an arbitrarily but fixed positive real number,*

(iii) $f(z) \to \mathscr{F}^{-1}[V] \in \mathfrak{H}'$ *in the strong (and weak) topology of* $\mathfrak{H}'$ *as* $y = \operatorname{Im} z \to 0$, $y \in C' \Subset C$.

The functions $f(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$ can be recovered as the (inverse) Fourier-Laplace transform of the constructed distribution $V \in H'_{C^*}(\mathbb{R}^n; O)$. This result is a generalization of the Paley-Wiener-Schwartz theorem for the setting of tempered ultrahyperfunctions.

Theorem 3.19 (Paley-Wiener-Schwartz-type Theorem [48])**.** *Let* $f(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$*, where* C *is an open convex cone. Then the distribution* $V \in H'_{C^*}(\mathbb{R}^n; O)$ *has a uniquely determined inverse Fourier-Laplace transform* $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi, z\rangle})$ *which is holomorphic in* $T(C')$ *and satisfies the estimate (3.6), with* $B[0; r]$ *replaced by* $B(0; r)$*.*

The following corollary is immediate from Theorem 3.19.

Corollary 3.20 ([46])**.** *Let* C^* *be a closed convex cone and* K *a convex compact set in* $\mathbb{R}^n$*. Define an indicator function* $h_{K,C^*}(y)$, $y \in \mathbb{R}^n$, *and an open convex cone* C_K *such that* $h_{K,C^*}(y) = \sup_{\xi\in C^*}\big|h_K(\xi)-\langle \xi, y\rangle\big|$ *and* $C_K = \big\{y \in \mathbb{R}^n \mid h_{K,C^*}(y)<\infty\big\}$*. Then the distribution* $V \in H'_{C^*}(\mathbb{R}^n; O)$ *has a uniquely determined inverse Fourier-Laplace transform* $f(z) = (2\pi)^{-n}(V, e^{-i\langle \xi, z\rangle})$ *which is holomorphic in the tube* $T(C'_K) = \mathbb{R}^n + iC'_K$*, and satisfies the following estimate, for a suitable* $K \subset \mathbb{R}^n$*,*

$$\big|f(z)\big| \leq \mathbf{C}(C')(1 + |z|)^N e^{h_{K,C^*}(y)} \ , \quad z \in T(C'_K; r) = \mathbb{R}^n + i\big(C'_K \setminus \big(C'_K \cap B(0; r)\big)\big) \tag{3.8}$$

where $C'_K \Subset C_K$*.*

The same proof as in Carmichael [41, Theorem 1, equation (4)] combined with the proofs of Theorems 3.18 and 3.19 shows that the following theorem is true.

Theorem 3.21. *Let* C *be an open convex cone, and let* $C' \Subset C$*. Let* $f(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$*. Then there exists a unique element* $V \in H'_{C^*}(\mathbb{R}^n; O)$ *such that*

$$f(z) = \mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} V\big] \ , \quad z \in T(C'; r) = \mathbb{R}^n + i\big(C' \setminus \big(C' \cap B(0; r)\big)\big) \ , \tag{3.9}$$

where (3.9) holds as an equality in $\mathfrak{H}'(T(O))$*.*

Remark 4. It is important to remark that in Theorems 3.18 and 3.19 we are considering the inverse Fourier-Laplace transform $f(z) = (2\pi)^{-n}\big\langle V, e^{-i\langle \xi, z\rangle}\big\rangle$, in opposition to the Fourier-Laplace transform used in the proof of Theorem 1 of Ref. [41]. In this case the proof of Theorem 3.21 is achieved if we consider ξ as belonging to the open half-space $\big\{\xi \in C^* \mid \langle \xi, y\rangle < 0\big\}$, for $y \in C' \setminus \big(C' \cap B(0; r)\big)$, since by hypothesis $f(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$. Then, from [55, Lemma 2, p.223] there is $\delta(C')$ such that for $y \in C' \setminus \big(C' \cap B(0; r)\big)$ implies $\langle \xi, y\rangle \leq -\delta(C')|\xi||y|$. This justifies the negative sign in (3.9).

In this point, we note the following fact important. Let $\mathfrak{H}'_C(T(O))$ denote the subset of $\mathfrak{H}'(T(O))$ defined by $\mathfrak{H}'_C(T(O)) = \{U \in \mathfrak{H}'(T(O)) \mid U = \mathscr{F}[V], V \in H'_{C^*}(\mathbb{R}^n; O)\}$. Then, by exactly the same arguments explained in [42, p.114], we have the following corollary of Theorems 3.18, 3.19 and 3.21.

Corollary 3.22. *Let C be an open convex cone. Then $\mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$ is algebraically isomorphic to both $H'_{C^*}(\mathbb{R}^n; O)$ and $\mathfrak{H}'_C(T(O))$.*

We finish this section with two results proved in Ref. [48], which will be used in the applications of Section 5.

Theorem 3.23 (Ultrahyperfunctional version of edge of the wedge theorem)**.** *Let C be an open cone of the form $C = C_1 \cup C_2$, where each C_j, $j = 1, 2$, is a proper open convex cone. Denote by $\boldsymbol{ch}(C)$ the convex hull of the cone C. Assume that the distributional boundary values of two holomorphic functions $f_j(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c_j}}$ $(j = 1, 2)$ agree, that is, $U = BV(f_1(z)) = BV(f_2(z))$, where $U \in \mathfrak{H}'$ in accordance with the Theorem 3.18. Then there exists $F(z) \in \mathscr{H}^{\boldsymbol{o}}_{\boldsymbol{ch(C)}}$ such that $F(z) = f_j(z)$ on the domain of definition of each $f_j(z)$, $j = 1, 2$.*

Theorem 3.24. *Let C be some open convex cone. Let $f(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$. If the boundary value $BV(f(z))$ of $f(z)$ in the sense of tempered ultrahyperfunctions vanishes, then the function $f(z)$ itself vanishes.*

4. Wightman Functionals for UHFNCQFT and Their Properties

According to Wightman, the conventional postulates of QFT can be fully reexpressed in terms of an equivalent set of properties of the vacuum expectation values of their ordinary field products, called Wightman distributions

$$\mathfrak{W}_m(f_1 \otimes \cdots \otimes f_m) \stackrel{\text{def}}{=} \langle \Omega_o \mid \Phi(f_1) \cdots \Phi(f_m) \mid \Omega_o \rangle \ , \tag{4.1}$$

where $(f_1 \otimes \cdots \otimes f_m) = f_1(x_1) \cdots f_m(x_m)$ is considered as an element of $\mathscr{S}(\mathbb{R}^{4m})$, and $\mid \Omega_o\rangle$ is the vacuum vector, unique vector time-translation invariant of the Hilbert space of states.

Remark 5. To keep things as simple as possible, we will assume that the Wightman distributions are “functions” $\mathfrak{W}_m(x_1, \ldots, x_m)$. The reader can easily supply the necessary test functions.

As a general rule, the continuous linear functionals $\mathfrak{W}_m(x_1, \ldots, x_m)$ are assumed to satisfy the following properties:

- $\mathsf{P_1}$ (Temperedness). The sequence of Wightman functions $\mathfrak{W}_m(x_1, \ldots, x_m)$ are tempered distributions in $\mathscr{S}'(\mathbb{R}^{4m})$, for all $m \geq 1$. This property is included in the list of properties for a QFT for technical reasons.

P_2 (Poincaré Invariance). Wightman functions are invariant under the Poincaré group

$$\mathfrak{W}_m(\Lambda x_1 + a, \ldots, \Lambda x_m + a) = \mathfrak{W}_m(x_1, \ldots, x_m) \ .$$

P_3 (Spectral Condition). The Fourier transforms of the Wightman functions have support in the region

$$\text{(SC)} \qquad \left\{ (p_1, \ldots, p_m) \in \mathbb{R}^{4m} \;\middle|\; \sum_{j=1}^{m} p_j = 0,\ \sum_{j=1}^{k} p_j \in \overline{V}_+,\ k = 1, \ldots, m-1 \right\} ,$$

where $\overline{V}_+ = \{(p^0, \boldsymbol{p}) \in \mathbb{R}^4 \mid p^2 \geq 0, p^0 \geq 0\}$ is the closed forward light cone.

P_4 (Local commutativity). This property has origin in the quantum principle that operator observables $\Phi(x)$ corresponding to independent measurements must comute.

$$\mathfrak{W}_m(x_1, \ldots, x_j, x_{j+1}, \ldots, x_m) = \mathfrak{W}_m(x_1, \ldots, x_{j+1}, x_j, \ldots, x_m) \ ,$$

if $(x_j - x_{j+1})^2 < 0$.

P_5 For any finite set $f_o, f_1, \ldots, f_N$ of test functions such that $f_o \in \mathbb{C}$, $f_j \in \mathscr{S}(\mathbb{R}^{4j})$ for $1 \leq j \leq N$, one has

$$\sum_{k,\ell=0}^{N} \mathfrak{W}_{k+\ell}(f_k^* \otimes f_\ell) \geq 0 \ .$$

P_6 (Hermiticity). A neutral scalar field must be real valued. This implies that

$$\mathfrak{W}_m(x_1, x_2, \ldots, x_{m-1}, x_m) = \overline{\mathfrak{W}_m(x_m, x_{m-1}, \ldots, x_1, x_2)} \ .$$

Generalizing these properties to NCQFT is not as simple, especially the Lorentz symmetry. For example, as already mentioned in the Introduction, the Lorentz symmetry is not preserved in NCQFT. Furthemore, the existence of hard infrared singularities in the non-planar sector of the theory can destroy the *tempered* nature of the Wightman functions. And more, how can the Property **P_4** be described in field theory with a fundamental length? In order to answer these questions, we shall assume a NCFT where the Wightman functionals fulfil a set of properties which actually will characterize a UHFNCQFT.

4.1. **Twisted Poincaré Symmetry.** In this paper, we will assume that our fields are transforming according to representations of the *twisted* Poincaré group [12, 13]. This formalism has the advantage of retaining the Wigner's notion of elementary particles.[2]

When referring to NCQFT one should have in mind the deformation of the ordinary product of fields. This deformation is performed in terms of the star product extended for noncoinciding points via the functorial relation [14]

$$\varphi(x_1)\star\cdots\star\varphi(x_n) = \prod_{i<j}\exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x_i^\mu}\otimes\frac{\partial}{\partial x_j^\nu}\right)\varphi(x_1)\cdots\varphi(x_n)\ . \tag{4.2}$$

For coinciding points $x_1 = x_2 = \cdots = x_n$ the product (4.2) becomes identical to the multiple Moyal $\star$-product. We shall consider NCQFT in the sense of a field theory on a non-commutative spacetime encoded by a Moyal product on the test function algebra.

Definition 4.1 (Vacuum Expectation Values of Fields [2])**.** *In a UHFNCQFT the Wightman functionals in $\mathscr{U}_c(\mathbb{R}^{4m})$, i.e., the m-points vacuum expectation values of fields operators are defined by*

$$\mathfrak{W}^\star_m(z_1,\ldots,z_m) \stackrel{\text{def}}{=} \langle\Omega_o \mid \Phi(z_1)\star\cdots\star\Phi(z_m)\mid\Omega_o\rangle\ . \tag{4.3}$$

Remark 6. The tempered ultrahyperfunctions $\mathfrak{W}^\star_m \in \mathscr{U}_c(\mathbb{R}^{4m})$ will be called *non-commutative* Wightman functions.

Remark 7. In [5] the Wightman functions were written as follows:

$$\mathfrak{W}^{\tilde\star}_m(z_1,\ldots,z_m) \stackrel{\text{def}}{=} \langle\Omega_o \mid \Phi(z_1)\tilde\star\cdots\tilde\star\,\Phi(z_m)\mid\Omega_o\rangle\ ,$$

where the meaning of $\tilde\star$ depends on the considered case. In particular, if $\tilde\star = 1$, we obtain the standard form $\mathfrak{W}_m(z_1,\ldots,z_m) = \langle\Omega_o \mid \Phi(z_1)\cdots\Phi(z_m)\mid\Omega_o\rangle$ adopted in [1], which corresponds to the commutative theory with the $SO(1,1)\times SO(2)$ invariance. On the other hand, if $\tilde\star = \star$, this choice corresponds to the Wightman functions introduced in [2]. In this case, the non-commutativity is manifested not only at coincident points but also in their neighborhood.

As a consequence of the twisted Poincaré covariance condition of the $\star$-product of fields [13], the non-commutative Wightman functions $\mathfrak{W}^\star_m(z_1,\ldots,z_m)\in\mathscr{U}_c(\mathbb{R}^{4m})$ satisfy the twisted Poincaré transformations (besides of the symmetry $SO(1,1)\times SO(2)$). Thus, we have the

Theorem 4.2. $\mathfrak{W}^\star_m(z_1,\ldots,z_m) = \mathfrak{W}^\star_m(\Lambda z_1 + a,\ldots,\Lambda z_m + a)$*, in the usual distributional sense.*

[2]Another approach where the full Poincaré group is preserved was proposed by Doplicher-Fredenhagen-Roberts [56].

4.2. **Domain of Analyticity of Non-Commutative Wightman Functions.** Since for non-commutative theories the group of translations is intact, the Wightman functions only depends on the $(m-1)$ coordinate differences as in the commutative case. Then, passing to the difference variables ζ_i, we obtain, symbolically, that

$$\mathfrak{W}^{\star}_m(z_1,\dots,z_m) = W^{\star}_m(\zeta_1,\dots,\zeta_{m-1})\ , \quad \zeta_j = z_j - z_{j+1}\ , \quad j=1,\dots,m-1\ .$$

Applying Corollary 3.20 to the ordinary Wightman functions $W_m(\zeta_1,\dots,\zeta_{m-1})$, we obtain the following important result:

Theorem 4.3. *The functions $W_{m-1}(\zeta_1,\dots,\zeta_{m-1})$ are holomorphic functions of $4(m-1)$ complex variables in a set which contains $\mathbb{R}^{4(m-1)} + V_+(\ell_{\theta_1},\dots,\ell_{\theta_{m-1}})$, where*

$$V_+(\ell_{\theta_1},\dots,\ell_{\theta_{m-1}}) = \Big\{(\eta_1,\dots,\eta_{m-1}) \in \mathbb{R}^{4(m-1)} \mid \eta_j = y_j + (\ell_{\theta_j},\mathbf{0}) \in V_+ + (\ell_{\theta_j},\mathbf{0})\Big\}\ ,$$

and satisfy the estimate

$$\big|W_{m-1}(\zeta_1,\dots,\zeta_{m-1})\big| \leq \mathsf{C}(V')\prod_{j=1}^{m-1}(1+|\zeta_j|)^N \exp\Big(h_{K,\overline{V}_+^{m-1}}(y_j)\Big)\ . \tag{4.4}$$

Proof. The first part of theorem follows immediately from Remark 2.18 in [46]. Thus we need only show that $W_{m-1}(\zeta_1,\dots,\zeta_{m-1})$ satisfies the estimate (4.4). But, this can be proved by using the Theorem 3.19 in order to show that the function $W_{m-1}(\zeta_1,\dots,\zeta_{j-1},\zeta',\zeta_{j+1},\dots,\zeta_{m-1})$ is a holomorphic function of ζ' alone, with the complex variables $\zeta_1,\dots,\zeta_{j-1},\zeta_{j+1},\dots,\zeta_{m-1}$ being kept fixed. Then, we apply this argument, in turn, to each variable ζ_j separately. □

Proposition 4.4. *In a UHFNCQFT the Wightman functionals in $\mathscr{U}_c(\mathbb{R}^{4(m-1)})$, i.e., the non-commutative Wightman functions involving the $\star$-product, $W^{\star}_{m-1}$, coincide with the standard Wightman functions W_{m-1}.*

Proof. By considering that in terms of complex variables

$$\prod_{i<j}\exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x_i^{\mu}}\otimes\frac{\partial}{\partial x_j^{\nu}}\right) = \prod_{i<j}\exp\left(\frac{1}{2}\theta^{\mu\nu}\frac{\partial}{\partial \zeta_i^{\mu}}\wedge\frac{\partial}{\partial \bar{\zeta}_j^{\nu}}\right)\ ,$$

and since the functions $W_m(\zeta_1,\dots,\zeta_{m-1})$ are holomorphic, then it follows that

$$W^{\star}_{m-1}(\zeta_1,\dots,\zeta_{m-1}) = W_{m-1}(\zeta_1,\dots,\zeta_{m-1})\ ,$$

and the proof is complete. □

Corollary 4.5. *The non-commutative Wightman functions $W^{\star}_{m-1}(\zeta_1,\dots,\zeta_{m-1})$ are holomorphic functions of $4(m-1)$ complex variables in a set which contains $\mathbb{R}^{4(m-1)} + V_+(\ell_{\theta_1},\dots,\ell_{\theta_{m-1}})$, and*

satisfy the estimate

$$\left|W^{\star}_{m-1}(\zeta_1,\ldots,\zeta_{m-1})\right| \leq \mathbf{C}(V') \prod_{j=1}^{m-1} (1+|\zeta_j|)^N \exp\Big(h_{K,\overline{V}_+^{m-1}}(y_j)\Big) \ .$$

It is suggestive to see that $W^{\star}_{m-1}$ has the same form as the standard form W_{m-1} in a UHFNCQFT. In light of Proposition 4.4, where we have as result that $\tilde{\star} = \star = 1$, we conjecture that the possibility of extending the axiomatic approach to the NCQFT in terms of tempered ultrahyperfunctions is *independent* of the concrete type of the $\tilde{\star}$-product (similar conclusion was obtained in [5]). In order to support this conjecture, in Section 5, we derive for the UHFNCQFT the validity of some important theorems. These include the existence of CPT symmetry and the connection between Spin and Statistics for UHFNCQFT, in the case of space-space non-commutativity. In what follows, we shall always refer to the functions $W^{\star}_{m-1}$ in order to include non-commutativity effects not only into the vacuum state, as it happens with the functions W_{m-1}.

4.3. **Extended Local Commutativity Condition.** The existence of a minimum length related to the scale of nonlocality ℓ_θ [14] renders impossible the preservation of the canonical commutation rules since those rules make sense only in the distance regions greater than ℓ_θ. Thus, in order to remedy this difficulty the local commutativity will be replaced by a distinguished localization property in the sense of Brüning-Nagamachi [46], called *extended local commutativity*. This property is defined as a continuity condition of the expectation values of the field commutators in a topology associated to a ℓ_θ-neighborhood of the light cone.

Let $|x|_1$ be the norm

$$|x|_1 = |x_0| + |\boldsymbol{x}| \ , \quad |\boldsymbol{x}| = \sqrt{\sum_{i=1}^{3} (x_j)^2} \ ,$$

for $x=(x_0,\boldsymbol{x}) \in \mathbb{R}^4$. Denote

$$L^\ell = \Big\{(x_1,x_2) \in \mathbb{R}^8 \mid |x_1 - x_2|_1 < \ell_\theta\Big\} \ .$$

Define the open set V_+ of all strictly time-like points in $\mathbb{R}^4$ by

$$V_+ = \Big\{x \in \mathbb{R}^4 \mid (x_0)^2 - \boldsymbol{x}^2 > 0\Big\} \ .$$

In order to prepare for the definition of the extended local commutativity, we shall consider functionals which are carried by sets close to $\mathbb{R}^4$ but not contained in $\mathbb{R}^4$. Denote by V^{ℓ_θ} the complex ℓ_θ-neighborhood of V_+

$$V^{\ell_\theta} = \Big\{z \in \mathbb{C}^4 \mid \exists\, x \in V_+, |\mathrm{Re}\, z - x| + |\mathrm{Im}\, z|_1 < \ell_\theta\Big\} \ .$$

Consider the set of all pairs of points in $\mathbb{C}^4$ whose difference belongs to the ℓ_θ-neighborhood,

$$M^{\ell_\theta} = \Big\{ (z_1, z_2) \in \mathbb{C}^8 \mid z_1 - z_2 \in V^{\ell_\theta} \Big\} \ ,$$

and introduce the space $\mathfrak{H}(M^{\ell_\theta})$ consisting of all holomorphic functions on M^{ℓ_θ}. Then, according to Brüning-Nagamachi [46], we formulate the axiom of extended local commutativity condition as follows.

Definition 4.6 (extended local commutativity condition)**.** *Let f, g be two test functions in $\mathfrak{H}(T(\mathbb{R}^4))$, then the fields $\Phi(f)$ and $\Phi(g)$ are said to commute for any relative spatial separation $\ell' > \ell_\theta$ of their arguments, if the functional*

$$\begin{aligned} \mathsf{F} &= \big\langle \Theta \mid \big[\varphi(f), \varphi(g)\big]_\star \mid \Psi \big\rangle \\ &= \big\langle \Theta \mid \big(\varphi(f) \star \varphi(g) - \varphi(g) \star \varphi(f)\big) \mid \Psi \big\rangle \ , \end{aligned} \tag{4.5}$$

is carried by the set $M^{\ell'} = \big\{ \big(z_1, z_2\big) \in \mathbb{C}^8 \mid z_1 - z_2 \in V^{\ell'} \big\}$, for any vectors $\Theta, \Psi \in D_0$, i.e., if the functional F can be extended to a continuous linear functional on $\mathfrak{H}(M^{\ell'})$.

The Definition 4.6 can be understood saying that two operators $\Phi(f)$ and $\Phi(g)$, at two distinct points of the non-commutative spacetime, can not be distinguished if the relative spatial distance between their arguments is less than ℓ_θ. In other words, in NCQFT the quantum fluctuations of the spacetime *operationally* prevent the exact localization of the events inside of the minimum area ℓ_θ^2. This area is interpreted as the minimum region which *observables* can be probed [57].

Moreover, it follows from the extended local commutativity condition and from the Propositions 4.3 and 4.4 in [46] that the functional $\mathsf{F} \in \mathscr{U}_c(\mathbb{R}^{4m})$ defined by

$$\mathsf{F} = \mathfrak{W}^\star_m(z_1, \ldots, z_j, z_{j+1}, \ldots, z_m) - \mathfrak{W}^\star_m(z_1, \ldots, z_{j+1}, z_j, \ldots, z_m) \ ,$$

for any $\ell' > \ell_\theta$, $m \geq 2$ and $j \in \{1, \ldots, m-1\}$, can be extended to a continuous linear functional on $\mathfrak{H}(M_j^{\ell'})$, with $M_j^{\ell'} = \Big\{ \big(z_1, \ldots, z_m\big) \in \mathbb{C}^{4m} \mid z_j - z_{j+1} \in V^{\ell'} \Big\}$.

4.4. **Properties of Non-Commutative Wightman Functions.** The analysis of the preceding results has shown that the sequence of vacuum expectation values of a NCQFT in terms of tempered ultrahyperfunctions satisfies a number of specific properties. We summarize these below:

- P_1' $\mathfrak{W}^\star_0 = 1$, $\mathfrak{W}^\star_m \in \mathscr{U}_c(\mathbb{R}^{4m})$ for $n \geq 1$, and $\mathfrak{W}^\star_m(f^*) = \overline{\mathfrak{W}^\star_m(f)}$, for all $f \in \mathfrak{H}(T(\mathbb{R}^{4m}))$, where $f^*(z_1, \ldots, z_m) = \overline{f(\bar{z}_1, \ldots, \bar{z}_m)}$.

- P_2' The Wightman functionals $\mathfrak{W}^\star_m$ are invariant under the *twisted* Poincaré group

P_3' Spectral condition. Since the Fourier transformation of tempered ultrahyperfunctions are distributions, the spectral condition is not so much different from that of Schwartz distributions. Thus, for every $m \in \mathbb{N}$, there is $\widehat{\mathfrak{W}}_m^\star \in H'_{V^*}(\mathbb{R}^{4m},\mathbb{R}^{4m})$ [46], where

$$H'_{V^*}(\mathbb{R}^{4m},\mathbb{R}^{4m}) = \Big\{ V \in H'(\mathbb{R}^{4m},\mathbb{R}^{4m}) \mid \operatorname{supp}(\widehat{\mathfrak{W}}_m^\star) \subset V^* \Big\} , \tag{4.6}$$

with V^* being the properly convex cone (SC) defined in P_3.

P_4' Extended local commutativity condition.

P_5' For any finite set $f_o, f_1, \ldots, f_N$ of test functions such that $f_o \in \mathbb{C}$, $f_j \in \mathfrak{H}(T(\mathbb{R}^{4j}))$ for $1 \leq j \leq N$, one has

$$\sum_{k,\ell=0}^{N} \mathfrak{W}_{k+\ell}^\star(f_k^* \otimes f_\ell) \geq 0 \ .$$

5. CPT, Spin-Statistics and All That in UHFNCQFT

In the preceding sections, we have defined what is meant by NCQFT in terms of tempered ultrahyperfunctions and assembled some tools to aid in the analysis of its structure. In this section, these are used to establish some important theorems as the celebrated CPT and spin-statistics theorems. The proof of these results as given in the literature [7]-[10] usually seem to rely on the local character of the distributions in an essential way. In the approach which we follow the apparent source of difficulties in proving these results is the fact that for functionals belonging to the space of tempered ultrahyperfunctions the standard notion of the localization principle breaks down.

Let Φ be a Hermitian scalar field. For this field, it is well-known that in terms of the Wightman functions, a necessary and sufficient condition for the existence of CPT theorem is given by:

$$\mathfrak{W}_m(x_1,\ldots,x_m) = \mathfrak{W}_m(-x_m,\ldots,-x_1) \ . \tag{5.1}$$

Under the usual temperedness assumption, the proof of the equality (5.1) as given by Jost [58] starts of the weak local commutativity (WLC) condition, namely under the condition that the vacuum expectation value of the commutator of n scalar fields vanishes outside the light cone, which in terms of Wightman functions takes the form

$$\mathfrak{W}_m(x_1,\ldots,x_m) - \mathfrak{W}_m(x_m,\ldots,x_1) = 0 \ , \quad \text{for} \quad x_j - x_{j+1} \in \mathscr{J}_m \ . \tag{5.2}$$

Jost's proof that the WLC condition (5.2) is equivalent to the CPT symmetry (5.1) one relies on the fact that the proper complex Lorentz group contains the total spacetime inversion. Therefore, the equality $\mathfrak{W}_n(x_m,\ldots,x_1) = \mathfrak{W}_n(-x_m,\ldots,-x_1)$ holds, taking in account the symmetry property $\mathscr{J}_m = -\mathscr{J}_m$ in whole extended analyticity domain, by the Bargman-Hall-Wightman (BHW)

theorem. In particular, the BHW theorem has been shown [46] to be applicable to domains of the form $\mathscr{T}_{m-1} = \mathbb{R}^{4(m-1)} + V_+(\ell'_1, \dots, \ell'_{m-1})$. Then, $W^\star_m(\zeta_1, \dots, \zeta_{m-1})$ can be extended to be a holomorphic function on the extended tube

$$\mathscr{T}^{\text{ext.}}_{m-1} = \Big\{(\Lambda\zeta_1, \dots, \Lambda\zeta_{m-1})) \mid (\zeta_1, \dots, \zeta_{m-1}) \in \mathscr{T}_{m-1}, \Lambda \in \mathscr{L}_+(\mathbb{C})\Big\} ,$$

which contains certain real points of type of the Jost points.

In order to prove that CPT theorem holds in NCQFT, an analogous of the WLC condition is now formulated:

Definition 5.1. *The non-commutative quantum field Φ defined on the test function space $\mathfrak{H}(T(\mathbb{R}^4))$ is said to satisfy the weak extended local commutativity (WELC) condition if the functional*

$$\mathsf{F} = \mathfrak{W}^\star_m(z_1, \dots, z_m) - \mathfrak{W}^\star_m(z_n, \dots, z_1) ,$$

is carried by set $M^{\ell'}_j = \Big\{\big(z_1, \dots, z_m\big) \in \mathbb{C}^{4m} \mid z_j - z_{j+1} \in V^{\ell'}\Big\}$.

The WELC condition takes the form $W^\star_m(\zeta_1, \dots, \zeta_{m-1}) - W^\star_m(-\zeta_{m-1}, \dots, -\zeta_1)$ in terms of the NC Wightman functions depending on the relative coordinates $\zeta_j = z_j - z_{j+1} \in V^{\ell'}$.

Proposition 5.2. *Consider $W^\star_m(\zeta_1, \dots, \zeta_{m-1})$ and $W^\star_m(-\zeta_{m-1}, \dots, -\zeta_1)$. Then*

$$W^\star_m(\zeta_1, \dots, \zeta_{m-1}) = W^\star_m(-\zeta_{m-1}, \dots, -\zeta_1) ,$$

on their respective domains of holomorphy.

Proof. The idea of the proof follows from the standard strategy. As in Ref. [7] suppose that $x_1, \dots, x_m$ are such that all the differences $x_i - x_j$ are space-like. Then $(z_1, \dots, z_m) \notin M^{\ell'}_j$. Hence,

$$W^\star_m(\zeta_1, \dots, \zeta_{m-1}) = W^\star_m(-\zeta_{m-1}, \dots, -\zeta_1)$$

by Definition 5.1. Now, our propose is to show that these are points of holomorphy of both functions. This is achieved applying the Edge of the Wedge theorem (Theorem 3.23). First, we note that $W^\star_m(\zeta_1, \dots, \zeta_{m-1})$ is holomorphic in $\mathbb{R}^{4(m-1)} + V_+(\ell'_1, \dots, \ell'_{m-1})$ by Corollary 4.5. Furthermore, the functions $W^\star_m(\zeta_1, \dots, \zeta_{m-1})$ and $W^\star_m(-\zeta_{m-1}, \dots, -\zeta_1)$ have boundary values which agree at totally space-like points in the sense of the strong topology of $\mathfrak{H}'$. Hence, by Theorem 3.23 $W^\star_m(-\zeta_{m-1}, \dots, -\zeta_1)$ is holomorphic at such points. □

Theorem 5.3 (CPT Theorem)**.** *A non-commutative scalar field theory symmetric under the CPT-operation Θ is equivalent to the WELC.*

Proof. The CPT invariance condition is derived by requiring that the CPT operator Θ be antiunitary – see [7]-[10]:

$$\langle \Theta\Xi \mid \Theta\Psi\rangle = \langle \Psi \mid \Xi\rangle . \tag{5.3}$$

This means that the CPT operator leaves invariant all transition probabilities of the theory. In the case of a NCFT, the operator Θ can be constructed in the ordinary way. Taking the vector states as $\langle \Xi \mid = \langle \Omega_o \mid$ and $\mid \Psi \rangle = \Phi(z_m) \star \cdots \star \Phi(z_1) \mid \Omega_o \rangle$ we shall express both sides of (5.3) in terms of NC Wightman functions. For the left-hand side of (5.3) we can use directly the CPT transformation properties of the field operators, which for a neutral scalar field is equal to $\Theta\Phi(z)\Theta^{-1} = \Phi(-z)$. Using the CPT-invariance of the vacuum state, $\Theta \mid \Omega_o \rangle = \mid \Omega_o \rangle$, the left-hand side of (5.3) becomes:

$$\begin{aligned} \langle \Theta\Xi \mid \Theta\Psi \rangle &= \langle \Theta\Omega_o \mid \Theta(\Phi(z_m) \star \cdots \star \Phi(z_1) \mid \Omega_o \rangle \\ &= \mathfrak{W}^{\star}_m(-z_m, \ldots, -z_1) \ . \end{aligned} \tag{5.4}$$

In order to express the right-hand side of (5.3), we take the Hermitian conjugates of the vectors $\mid \Psi \rangle$ and $\langle \Xi \mid$, to obtain:

$$\langle \Psi \mid \Xi \rangle = \mathfrak{W}^{\star}_m(z_1, \ldots, z_m) \ . \tag{5.5}$$

Putting together (5.3) with (5.4) and (5.5), we obtain the CPT invariance condition in terms of NC Wightman functions as

$$\mathfrak{W}^{\star}_m(z_1, \ldots, z_m) = \mathfrak{W}^{\star}_m(-z_n, \ldots, -z_1) \ ,$$

which in terms of the NC Wightman functions depending on the relative coordinates ζ_j reads

$$W^{\star}_m(\zeta_1, \ldots, \zeta_m) = W^{\star}_m(\zeta_{m-1}, \ldots, \zeta_1) \ , \tag{5.6}$$

Then, without giving more details, it should be clear from the Proposition 5.2 that the arguments of Chap. V of Ref. [8] apply in our case. Hence, the CPT theorem continues to hold in UHFNCQFT. □

As it is well-known, the Borchers class of a quantum field is a direct consequence of the CPT theorem. Thus, we have the

Theorem 5.4 (Borchers class of quantum fields for a NCQFT)**.** *Suppose Φ is a field satisfying the assumptions of Theorem 5.3 and Θ is the corresponding CPT-symmetry operator. Suppose ψ is another field transforming under the same representation of the* **twisted** *Poincaré group, with the same domain of definition. Suppose that the functional $\langle \Omega_o \mid \Phi(z_1) \star \cdots \star \Phi(z_j) \star \boldsymbol{\psi}(z) \star \Phi(z_{j+1}) \star \cdots \star \Phi(z_m) \mid \Omega_o \rangle - \langle \Omega_o \mid \Phi(z_m) \star \cdots \star \Phi(z_{j+1}) \star \boldsymbol{\psi}(z) \star \Phi(z_j) \star \cdots \star \Phi(z_1) \mid \Omega_o \rangle$ is carried by $M^{\ell'}_j = \Big\{ \big(z_1, \ldots, z_{m+1}\big) \in \mathbb{C}^{4(m+1)} \mid z_j - z_{j+1} \in V^{\ell'} \Big\}$. Then Θ implements the CPT symmetry for $\boldsymbol{\psi}$ as well and the fields $\Phi, \boldsymbol{\psi}$ satisfy the weak extended local commutativity condition.*

Proof. The proof is similar to the proof of Theorem 3.4 of Ref. [4]. □

Corollary 5.5 (Transitivity of the WELC)**.** *The weak relative extended local commutativity property is transitive in the sense that if each of the fields $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2$ satisfies the assumptions of Theorem 5.4, then there is a CPT-symmetry operator common to the fields $\{\Phi, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2\}$ and by Theorem 5.3, the weak relative extended local commutativity condition is satisfied not only for $\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2\}$ but also for $\{\Phi, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2\}$.*

Theorem 5.6 (Spin-Statistics Theorem)**.** *Suppose that Φ and its Hermitian conjugate Φ^* satisfy the WELC with the "wrong" connection of spin and statistics. Then $\Phi(x)\Omega_o = \Phi^*(x)\Omega_o = 0$.*

Proof. The arguments of the standard proof apply [7], since the properties of Lorentz group representations, existence of Jost points and the analyticity properties of NC Wightman functions are also available in UHFNCQFT. □

We complete this section with a of the most important results of the axiomatic approach: the Reconstruction theorem. Based in our analysis, we have the following

Theorem 5.7 (Reconstruction theorem to UHFNCQFT)**.** *Suppose that the hypotheses of Theorem 5.1 in* [46] *hold except that instead of the sequence $\{\mathfrak{W}_m\}_{m\in\mathbb{N}}$ and of the conditions $(R0)-(R5)$, we have the sequence $\{\mathfrak{W}^{\star}_m\}_{m\in\mathbb{N}}$ and the conditions $\mathbf{P'_1}-\mathbf{P'_5}$. Then the conclusions of Theorem 5.1 in* [46] *again hold.*

6. Concluding Remarks

In the present paper, we extend the Wightman axiomatic approach to NCQFT in terms of tempered ultrahyperfunctions. An important hint in favor of this approach comes from the fact that the class of UHFNCQFT allows for the possibility that the off-mass-shell amplitudes can grow at large energies faster than any polynomial (such behavior is not possible if fields are assumed to be tempered only). This is relevant since NCQFT stands as an intermediate framework between string theory and the usual quantum field theory. Here, we restrict ourselves to the simplest case, that of a single, scalar, Hermitian field $\Phi(x)$ associated with spinless particles of mass $m > 0$. Some results of the ordinary QFT, the existence of the symmetry CPT and of the Spin-Statistics connection were proved to hold, if we replace the local commutativity by an *extended local commutativity* in the sense of Brüning-Nagamachi [46]. We assume (implicitly) the case of a theory with space-space non-commutativity ($\theta_{0i} = 0$). There is still a number of important questions to be studied based on the ideas of this paper, such as the existence of the S-matrix, a representation of the Jost-Lehmann-Dyson-type, the Reeh-Schlieder property and so on. Furthemore, as it was pointed out in [1], for gauge theories, in particular the non-commutative QED (NCQED), the questions associated to the Wightman axioms and their consequences are more involved due to the UV/IR mixing. As said at the beginning, the existence of hard infrared singularities in the non-planar sector of the theory, induced by uncancelled quadratic ultraviolet divergences, can result in one

kind of problem: they can destroy the *tempered* nature of the Wightman functions. This result reinforces the hypothesis that the infrared issue in NCFT must be dealt with another approach. In this case the ultrahyperfunctional approach to NCQFT could be an interesting step in order to resolve the problem of the UV/IR mixing in NCFT. This topic is under investigation.[3] We hope to report our conclusions on this issue in a forthcoming paper.

As a last remark, we note the result obtained in [15] where has been showed that the star commutator of $:\ \phi(x) \star \phi(y)\ :$ and $:\ \phi(y) \star \phi(x)\ :$ does not obey the microcausality even for the case in which $\theta_{0i} = 0$. However, we see that this is not the case here. The condition of extended local commutativity being defined as a continuity condition of the expectation values of the field commutators in a topology associated to a complex neighborhood of the light cone, it is not applied to the tempered fields. Hence, for NCQFT in terms of tempered ultrahyperfunctions no violation of Einstein's causality is ever involved.

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